Exercise 26

Solve the differential equation using the method of variation of parameters.

$$y'' + 3y' + 2y = \sin(e^x)$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 3y_c' + 2y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2e^{rx}$$

Plug these formulas into equation (1).

$$r^2e^{rx} + 3(re^{rx}) + 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 3r + 2 = 0$$

Solve for r.

$$(r+2)(r+1) = 0$$

$$r = \{-2, -1\}$$

Two solutions to the ODE are e^{-2x} and e^{-x}. By the principle of superposition, then,

$$y_c(x) = C_1e^{-2x} + C_2e^{-x}.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + 3y_p' + 2y_p = \sin(e^x) \tag{2}$$

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_1(x)e^{-2x} + C_2(x)e^{-x}$$

Differentiate it with respect to x.

$$y_p' = C_1'(x)e^{-2x} + C_2'(x)e^{-x} - 2C_1(x)e^{-2x} - C_2(x)e^{-x}$$

If we set

$$C_1'(x)e^{-2x} + C_2'(x)e^{-x} = 0, \tag{3}$$

then

$$y_p' = -2C_1(x)e^{-2x} - C_2(x)e^{-x}.$$

Differentiate it with respect to x once more.

$$y_p'' = -2C_1'(x)e^{-2x} - C_2'(x)e^{-x} + 4C_1(x)e^{-2x} + C_2(x)e^{-x}$$

Substitute these formulas into equation (2).

$$[-2C_1'(x)e^{-2x} - C_2'(x)e^{-x} + 4C_1(x)e^{-2x} + C_2(x)e^{-x}] + 3[-2C_1(x)e^{-2x} - C_2(x)e^{-x}] + 2[C_1(x)e^{-2x} + C_2(x)e^{-x}] = \sin(e^x)$$

Simplify the result.

$$-2C_1'(x)e^{-2x} - C_2'(x)e^{-x} = \sin(e^x) \tag{4}$$

Add the respective sides of equations (3) and (4) to eliminate $C_2'(x)$.

$$-C_1'(x)e^{-2x} = \sin(e^x)$$

Solve for $C_1'(x)$.

$$C_1'(x) = -e^{2x}\sin(e^x)$$

Integrate this result to get $C_1(x)$, setting the integration constant to zero.

$$\begin{aligned} C_1(x) &= \int^x C_1'(w)\,dw \\ &= -\int^x e^{2w}\sin(e^w)\,dw \\ &= -\int^{e^x} u\sin u\,du \\ &= -(\sin u - u\cos u)\Big|^{e^x} \\ &= (u\cos u - \sin u)\Big|^{e^x} \\ &= e^x\cos(e^x) - \sin(e^x) \end{aligned}$$

Solve equation (3) for $C_2'(x)$.

$$\begin{aligned} C_2'(x) &= -C_1'(x)e^{-x} \\ &= -[-e^{2x}\sin(e^x)]e^{-x} \\ &= e^x\sin(e^x) \end{aligned}$$

Integrate this result to get $C_2(x)$, setting the integration constant to zero.

$$\begin{aligned} C_2(x) &= \int^x C_2'(w)\,dw \\ &= \int^x e^w \sin(e^w)\,dw \\ &= \int^{e^x} \sin u\,du \\ &= -\cos u\Big|^{e^x} \\ &= -\cos(e^x) \end{aligned}$$

Therefore,

$$\begin{aligned} y_p &= C_1(x)e^{-2x} + C_2(x)e^{-x} \\ &= [e^x\cos(e^x) - \sin(e^x)]e^{-2x} + [-\cos(e^x)]e^{-x} \\ &= -e^{-2x}\sin(e^x), \end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1e^{-2x} + C_2e^{-x} - e^{-2x}\sin(e^x), \end{aligned}$$

where C_1 and C_2 are arbitrary constants.